Upper Bound For Matrix Operators On Some Sequence Spaces

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Intisari

Di dalam paper ini, akan didiskusikan masalah pencarian batas atas dan norma operator matrik Hausdorff pada beberapa ruang barisan.

Kata kunci: *norma-F, fungsi-ϕ, matriks Hausdorff, batas atas.*

Abstract

In this paper, we considered the problem of finding the upper bound and the norm of the Hausdorff matrix operator on some sequence spaces.

Keywords: *F-norm, ϕ-function, Hausdorff matrix, upper bound.*

1. Preliminaries and Some Basic Notions

Operator theory plays an important role in both pure and applied mathematics. Therefore, it always receives a lot of attention from mathematicians from those areas. In this paper, we discuss about the norm of a certain matrix operator on a certain sequence space. The key references are Jameson and Lashkaripour [2000], [2002], Lashkaripour [2002],[2004],[2005], and Pecari et.al [2001].

In this section, we give some basic notions. As usual, $\mathbb{R}$ and $\mathbb{N}$ denote the real and natural numbers system, respectively. $\mathbb{R}^+$ denotes the collection of all positive real numbers. The collection of all sequences in $\mathbb{R}$ will be denoted by $\mathcal{S}$.

Let $X \subset \mathcal{S}$ be a linear space over $\mathbb{R}$. A function $\| \; \| : X \to \mathbb{R}$ is called an *F-norm* if it satisfies

(i) $\|x\| \geq 0$ for every $x \in X$,

$\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$, and

(iii) if $\{x_n\} \subset X$ is a sequence such that $\lim\limits_{n\to\infty} \|x_n - x\| = 0$ for some $x \in X$, and $\{a_n\}$ is a sequence of real numbers which converges to some $a \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \|a_n x_n - ax\| = 0$.

The linear space X equipped with the F-norm $\| \ \|$, *denoted* $(X, \| \ \|)$, *is called an F-normed space. When the F-norm* $\| \ \|$ *has been explicitly known, we write X instead of* $(X, \| \ \|)$. *An F-normed space is said to be complete if every Cauchy sequence in the space is convergent. A complete F-normed space is called a Frèchet space or shortly an F-space.*

A function $\phi : \mathbb{R} \to \mathbb{R}$ is called a ϕ*-function* if it satisfies

(i) $\phi(x) = 0 \Leftrightarrow x = 0$,
(ii) $\phi(-x) = \phi(x)$, for every $x \in \mathbb{R}$,
(iii) ϕ is increasing on $\mathbb{R}^+$,
(iv) ϕ is continuous on $\mathbb{R}$, and
(v) $\lim_{x\to\infty} \phi(x) = \infty$.

A ϕ*-function* ϕ *is said to satisfy a* δ_2*-condition if there exists a real number* $M > 0$ *such that* $\phi(2x) \le M\phi(x)$ *for every* $x \ge 0$. *For any sequence of positive numbers* $v = \{v_n\}$ *and* ϕ*-function* ϕ *that satisfies* δ_2*-condition, we define*

$$l_\phi = \left\{ \{x_n\} \in \mathcal{S} : \sum_{n=1}^{\infty} \phi(x_n) < \infty \right\},$$

$$l_\phi(v) = \left\{ \{x_n\} \in \mathcal{S} : \sum_{n=1}^{\infty} v_n . \phi(x_n) < \infty \right\}.$$

We observe that l_ϕ and $l_\phi(v)$ *are complete F-norm spaces with respect to* $\|.\|_\phi$ *and* $\|.\|_{\phi,v}$, *respectively, where*

$$\|x\|_\phi = \sum_{n=1}^{\infty} \phi(x_n) \quad \text{and} \quad \|x\|_{\phi,v} = \sum_{n=1}^{\infty} v_n . \phi(x_n).$$

In case, $\phi(t) = |t|^p$, $1 \le p < \infty$, we write $l_p(v)$ instead of $l_\phi(v)$.

Let $w = \{w_n\}$ be a decreasing positive sequence of real numbers such that $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. We define

$$d(w,p) = \left\{ x = \{x_n\} : \sum_{n=1}^{\infty} w_n \left(x_n^*\right)^p < \infty \right\}$$

where $\{x_n^*\}$ *is a decreasing sequence which can be found by rearranging* $\{|x_n|\}$. *It can be shown that* $d(w,p)$ *is a space of all sequences with finitely non-zero elements. Further,* $d(w,p)$ *is an F-normed space with respect to*

$$\|x\|_{d(w,p)} = \|x^*\|_{w,p}.$$

2. Matrix Operators

Let $\{a_n\}$ be a sequence of real numbers with $a_1 = 1$. For any $n \in \mathcal{N} \cup \{0\}$, we define the operator Δ^n as follows

$$\Delta^0 a_k = a_k,\ \Delta^1 a_k = a_k - a_{k+1}, \text{ and}$$

$$\Delta^n a_k = \Delta^{n-1}(\Delta^1 a_k),\ n = 2,3,4,\dots$$

Further, the matrix $H = (h_{ij})$*, where*

$$h_{ij} = \begin{cases} C_{j-1}^{i-1}.\Delta^{i-j} a_j &, \quad 1 \le j \le i \\ 0 &, \quad j > i \end{cases}$$

is called the Hausdorff matrix.

Let μ be a probability measure on $[0,1]$. For any $n \in \mathcal{N}$, we define the sequence $\{a_n\}$ by

$$a_n = \int_0^1 x^{n-1} d\mu(x), \qquad n = 1,2,3,\dots$$

then we get the Hausdorff matrix $H(\mu) = (h_{ij})$*, with*

$$h_{ij} = \begin{cases} C_{j-1}^{i-1}.\int_0^1 x^{j-1}(1-x)^{i-j}\, d\mu(x) &, \quad 1 \le j \le i \\ 0 &, \quad j > i \end{cases}$$

The followings are some kind of Hausdorff matrices:

1. $C(\alpha) = H(\mu_\alpha)$, where $d\mu_\alpha(t) = \alpha(1-t)^{\alpha-1}$,

2. $H_0(\alpha) = H(\mu_\alpha)$, where $d\mu_\alpha(t) = \dfrac{|\log t|^{\alpha-1}}{\Gamma(\alpha)} dt$, and

3. $G(\alpha) = H(\mu_\alpha)$, where $d\mu_\alpha(t) = \alpha t^{\alpha-1} dt$,

where $\alpha > 0$ *is any real number. The matrices* $C(\alpha), H_0(\alpha)$, and $G(\alpha)$ *are called a Cesaro, Holder, and Gamma matrix respectively.*

Let $v = \{v_n\}$ and $w = \{w_n\}$ be sequences of positive numbers. We consider the matrix operator $A : l_\phi(v) \to l_\phi(w)$

$$Ax = y = \{y_n\},$$

$$y_n = \sum_{j=1}^{\infty} a_{n,j} x_j .$$

The norm of A is given by

$$\|A\| = \sup\left\{\|Ax\|_{\phi,w} : x \in l_\phi(v),\ \|x\|_{\phi,v} \le 1\right\}.$$

We observe the following theorem.

Theorem 2.1 *Let $w = \{w_n\}$ be a decreasing sequence of positive real numbers. If the Hausdorff matrix operator $H(\mu)$ maps the space $l_\phi(w)$ into itself, then*

$$\|Hx\|_{\phi,w} \le \sup_{k \le n} \frac{w_n}{w_k} \|x\|_{\phi,w}$$

Proof: For simplicity, we write $H(\mu) = H$. Take any $x \in l_\phi(w)$, then

$$\begin{aligned}\|Hx\|_{\phi,w} &= \sum_{i=1}^{\infty} w_i .\phi\left(\sum_{j=1}^{i} C_{j-1}^{i-1}.\left(\int_0^1 t^{j-1}(1-t)^{i-j} d\mu(t)\right) x_j\right) \\ &\le \sum_{i=1}^{\infty} w_i \sum_{j=1}^{\infty} \phi(x_j) \\ &\le \sup_{k \le n} \frac{w_n}{w_k} \sum_{j=1}^{\infty} w_j .\phi(x_j) = \sup_{k \le n} \frac{w_n}{w_k} .\|x\|_{\phi,w}.\end{aligned}$$

As a straight consequence, we then have the following corollary.

Corollary 2.2 *If the Hausdorff matrix operator $H(\mu)$ maps the space l_ϕ into itself, then*

$$\|H\|_{\phi,1} \le 1.$$

In case, the ϕ-function ϕ is of the form $\phi(x) = |x|^p$, $1 < p < \infty$, then we get inequalities for the Hausdorff matrix operator *H.*

Theorem 2.3 *Let $v = \{v_n\}$ and $w = \{w_n\}$ be decreasing sequences of positive numbers, with $v_1 = 1$. If the Hausdorff matrix operator $H(\mu)$ maps $l_p(v)$ into $l_p(w)$, $1 < p < \infty$, then*

$$\left(\inf \frac{w_n}{v_n}\right)^{1/p} \int_0^1 t^{-1/p} d\mu(t) \le \|H\| \le \left(\sup \frac{w_n}{v_n}\right)^{1/p} \int_0^1 t^{-1/p} d\mu(t).$$

Proof: We write $H(\mu) = H$ for the simplicity. Let $x \in l_p(v)$, then

$$\|Hx\|_{p,w}^{p} = \sum_{i=1}^{\infty} w_i . \left(\sum_{j=1}^{i} C_{j-1}^{i-1} . \left(\int_0^1 t^{j-1}(1-t)^{i-j} d\mu(t) \right) x_j \right)^p$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} w_j . \left(C_{j-1}^{i-1} \left(\int_0^1 t^{j-1}(1-t)^{i-j} d\mu(t) \right) . x_j \right)^p$$

$$\leq \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{i=1}^{\infty} \frac{w_i}{v_i} . v_i . x_i^p$$

$$\leq \sup \frac{w_n}{v_n} . \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{i=1}^{\infty} v_i . x_i^p$$

$$= \sup \frac{w_n}{v_n} . \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \|x\|_{v,p}^{p} .$$

These prove the right hand side of the inequality. Further, we are going to prove the left hand side of the inequality.

Let $0 < \delta < \dfrac{1}{p}$, $x_n = (n)^{-(1/p)-\delta}$, and $\varepsilon \in (0,1)$. It is clear that $\{x_n\} \in l_p$. Since $0 < v_n \leq 1$ for every $n \in \mathbb{N}$, then $\{x_n\} \in l_p(v)$. Take α and N such that

$$\left(1 + \frac{1}{\alpha} \right)^{-2/p} > 1 + \varepsilon,$$

$$\int_{\alpha/n}^{1} t^{-1/p} d\mu(t) > (1-\varepsilon) \int_0^1 t^{-1/p} d\mu(t), \quad n \geq N, \text{ and}$$

$$\sum_{k=N}^{\infty} w_k x_k^p > (1-\varepsilon) \sum_{k=1}^{\infty} w_k x_k^p ,$$

then

$$(Hx)_n = \sum_{k=1}^{n} C_{k-1}^{n-1} \left(\int_0^1 t^{k-1}(1-t)^{n-k} d\mu(t) \right) x_k$$

$$\geq (1-\varepsilon)^2 x_n . \int_0^1 t^{-1/p} d\mu(t), \qquad n \geq N.$$

Hence

$$w_n^{1/p} (Hx)_n \geq (1-\varepsilon)^2 w_n^{1/p} x_n . \int_0^1 t^{-1/p} d\mu(t), \qquad n \geq N.$$

Further,

$$
\begin{aligned}
\|Hx\|_{p,w}^{p} &= \sum_{n=N}^{\infty} w_n (Hx)_n^p \\
&\geq (1-\varepsilon)^{2p} . \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{n=1}^{\infty} w_n x_n^p \\
&\geq (1-\varepsilon)^{2p+1} . \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{n=1}^{\infty} w_n x_n^p \\
&\geq (1-\varepsilon)^{2p+1} . \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \sum_{n=1}^{\infty} \frac{w_n}{v_n} v_n x_n^p \\
&\geq \inf \frac{w_n}{v_n} (1-\varepsilon)^{2p+1} . \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \|x\|_{v,p}^{p} .
\end{aligned}
$$

These implies

$$
\|Hx\|_{p,w}^{p} \geq \inf \frac{w_n}{v_n} . \left(\int_0^1 t^{-1/p} d\mu(t) \right)^p \|x\|_{v,p}^{p} .
$$

If in the Theorem 2.4, we take $v_n = w_n$ for every n, then we get the following corollaries.

Corollary 2.5 If the Hausdorff matrix $H(\mu)$ maps the space $l_p(w)$ into itself, then

$$
\|H\|_{p,w} = \int_0^1 t^{-1/p} d\mu(t) .
$$

Corollary 2.6 Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If the matrices $C(\alpha)$, $H_0(\alpha)$, and $G(\alpha)$ map the space $l_p(w)$ into itself, then

$$
\|C(\alpha)\|_{w,p} = \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)}, \qquad \alpha > 0
$$

$$
\|H_0(\alpha)\|_{w,p} = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{-1/p} |\log t|^{\alpha-1} dt, \qquad \alpha > 0
$$

$$
\|G(\alpha)\|_{w,p} = \frac{\alpha p}{\alpha p - 1}, \qquad \alpha p > 1.
$$

Let $w = \{w_n\}$ be a monoton decreasing sequence of positive real numbers such that $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. We define

$$d(w,p) = \left\{ x = \{x_n\} : \sum_{n=1}^{\infty} w_n \left(x_n^*\right)^p < \infty \right\}$$

where $\{x_n^*\}$ is a monoton decreasing sequence found by rearranging the sequence $\{|x_n|\}$. It can be proved that $d(w,p)$ is a space that its members are all finite sequences. Further, $d(w,p)$ is an F-normed space with respect to

$$\|x\|_{d(w,p)} = \left\|x^*\right\|_{w,p}.$$

Lemma 2.7 Let $p \geq 1$ and $A = (a_{i,j})$ be the operator on $d(w,p)$ that satisfies

(i) $a_{i,j} \geq 0$ for every i, j, and

(ii) $\sum_{i\in M} \sum_{j\in K} a_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}$ for every subset $M, K \subset \mathbb{N}$ that consists of m, n elements, respectively.

Then for every non negative elemen $x \in d(w,p)$, we have

$$\|Ax\|_{d(w,p)} \leq \left\|Ax^*\right\|_{d(w,p)}.$$

Proof: See Lashkaripour R. [2002].

Lemma 2.8 Let $p \geq 1$ and $A = (a_{ij})$ be an operator from $d(w,p)$ into itself such that $a_{ij} \geq 0$ for every i and j. If for every $x \in d(w,p)$,

$$Ax = \left(\sum_{j=1}^{\infty} a_{ij} x_j \right)^t$$

then the following statements are equivalent.

(a) $y_1 \geq y_2 \geq ... \geq 0$ whenever $x_1 \geq x_2 \geq ... \geq 0$.

(b) $r_{in} = \sum_{j=1}^{n} a_{ij}$ is a sequence such that $r_{(i+1)n} \leq r_{in}$ for every n.

Proof:

$(a) \Rightarrow (b)$**:** Let $x \in d(w,p)$ be an arbitrary, then $x = (x_1, x_2, ..., x_n, 0,0,...)$ for some $n \in \mathcal{N}$. If $e_k = (0,...0,1,0,0,...)$, that is a sequence with the k^{th}-coordinate is equal to 1 and the others are 0, then $x = \sum_{k=1}^{n} x_k e_k$. Further, by the hypothesis we have

$$0 \le y_i - y_{i+1} = \sum_{j=1}^{n} (a_{ij} - a_{(i+1)j}) x_j$$
.

$(b) \Rightarrow (a)$: If $x \in d(w,p)$, then $x = (x_1, x_2, ..., x_n, 0,0,...)$ for some $n \in \mathcal{N}$. For any *i*, we have

$$y_i = \sum_{j=1}^{n} a_{i,j} x_j = r_{i,1} x_1 + (r_{i,2} - r_{i,1}) x_2 + ... + (r_{i,n} - r_{i,n-1}) x_n$$

$$= r_{i,1}(x_2 - x_1) + r_{i,2}(x_3 - x_2) + ... + r_{i,n}(x_n - x_{n-1}).$$

Hence, $y_i \ge y_{i+1} \ge 0$ whenever $x_1 \ge x_2 \ge ... \ge 0$.

Let $H(\mu)$ be a Hausdorff matrix such that $\sum_{i \in M} \sum_{j \in K} a_{ij} \le \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$ for any subset $M, K \subset \mathcal{N}$, which consist of m, n elements, respectively. Following Lemma 2.7 and Lemma 2.8, then for any non negative decreasing sequence *x* we have

$$\|Hx\|_{d(w,p)} = \|Hx\|_{w,p}.$$

Further, by using Theorem 2.4, we have the following theorems.

Theorem 2.9 Let $p > 1$ and $H(\mu)$ be a Hausdorff matrix operator such that $\sum_{i \in M} \sum_{j \in K} a_{ij} \le \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$ for any subsets $M, K \subset \mathcal{N}$, which consist of m, n elements, respectively. Then $H(\mu)$ maps $d(w,p)$ into itself and

$$\|H\|_{d(w,p)} = \int_0^1 t^{-1/p} d\mu(p)$$
.

Theorem 2.10 Let $A = (a_{ij})$ be a matrix that satisfies the conditions (i) and (ii) in Lemma 2.7 and $\sum_{i=1}^{\infty} w_i . a_{i1}$ be convergent. If $\{v_n\}$ is a sequence such that

$$\sup \frac{S_n}{V_n} < \infty$$

where $S_n = \sum_{k=1}^{n} s_k$, $s_n = \sum_{k=1}^{\infty} w_k . a_{kn}$, and $V_n = \sum_{k=1}^{n} v_k$, then A is a bounded linear operator from $d(v,1)$ into $d(w,1)$ and

$$\|A\|_{v,w,1} = \sup \frac{S_n}{V_n}.$$

Proof: Let $x \in d(v,1)$ be sequence such that $x_1 \geq x_2 \geq ... \geq 0$. If $M = \sup \frac{S_n}{V_n}$, then

$$\|Ax\|_{w,1} = \sum_{i=1}^{\infty} w_i \sum_{j=1}^{\infty} a_{i,j} x_j = \sum_{j=1}^{\infty} s_j x_j$$

$$= \sum_{j=1}^{\infty} S_j \left(x_j - x_{j+1}\right) \leq M \sum_{j=1}^{\infty} V_j \left(x_j - x_{j+1}\right).$$

Since

$$\|x\|_{v,1} = \sum_{j=1}^{\infty} v_j x_j = \sum_{j=1}^{\infty} V_j \left(x_j - x_{j+1}\right)$$

then

$$\|Ax\|_{w,1} \leq M \|x\|_{v,1}.$$

This implies $\|A\|_{v,w,1} \leq M$.

Further, by letting $x_1 = x_2 = ... = x_n = 1$ and $x_{n+k} = 0$ for every $k \in N$, then we have

$$\|x\|_{v,1} = V_n \text{ and } \|Ax\|_{w,1} = S_n.$$

So, $\|A\|_{v,w,1} = M$.

3. Concluding Remarks

In this paper, we have succesfully constructed the sequence spaces $l_\phi(v)$ and $d(v,\phi)$, which is an *F*-space, respectively. Further, $d(v,\phi)$ is a sequence space where all of its elements are finite sequences. By restricting the function ϕ of the form $\phi(t)=|t|^p, 1\le p<\infty$, then we can formulate the upper bound and norm of certain matrix operator on $l_p(v)$ and $d(v,p)$. The works will be continued for matrix operators act on $l_\phi(v)$ and $d(v,\phi)$.

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